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Adams operations, algebras up to homotopy and cyclic homology

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Abstract

We define and study a new A_∞-algebra structure on the cyclic bar complex of a commutative algebra. Unlike the Getzler–Jones' one, this new structure is compatible with the Adams operations and provides therefore the right setting for studying operations on cyclic homology theories. Its construction relies on the properties of some formal polytopes, which should be thought of as intermediates between Stasheff polytopes and models for cyclic homology operations.

0. Introduction

Motivated by index theorems and two-dimensional topological field theory, Getzler and Jones emphasized that one should, as far as possible, handle constructions on cyclic homology theories at the chain level [4]. More precisely, they pointed out the striking role of Stasheff's A_∞-algebra structures in this setting (see also [2]). For example, the Hood–Jones product in negative cyclic homology, which was first constructed by means of acyclic models arguments, is best understood by constructing an A_∞-algebra structure on the cyclic bar complexes [6, 3].

We wish to define a structure on the negative cyclic homology of a commutative algebra, at the chain level, corresponding to the λ-ring structure of the higher K-theory. The Getzler–Jones A_∞-products are not suitable for that purpose since they behave badly with respect to the Feigin–Tsygan and Loday's Adams operations. That is why we construct a new A_∞-algebra structure on the cyclic bar complexes, which induces the usual products on the various cyclic homology theories. Adams operations are A_∞-algebra endomorphisms for this new A_∞-algebra structure, which therefore should be "the right one" for defining and studying operations on cyclic homology theories at the chain level.

This construction avoids direct combinatorial computations. It relies on geometrical ideas and extends to the cyclic setting the techniques which were developed in [10] for Hochschild homology.

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In particular, we construct some formal polytopes, which should be thought of as intermediates between Stasheff polytopes [12] and models for cyclic homology operations.

The first section is devoted to the study of these polytopes and introduces the two fundamental operations: higher twisted cartesian products and their linearization.

The second one focuses on the construction of a product and of "higher homotopies" on the cyclic bar complex of a commutative algebra over the rationals. We show that Adams operations are endomorphisms of the corresponding A_∞-algebra structure.

1. Twisted cartesian products

Let us first recall a few definitions and results concerning polytope groups (see [5, 10] for further details). In this section, vector space means real vector space and S_n denotes the symmetric group of order n. The field of real (resp. rational) numbers is written $\mathbb{R}$ (resp. $\mathbb{Q}$). The ring of integers (resp. the set of positive integers) is written $\mathbb{Z}$ (resp. $\mathbb{N}$). A convex polytope is the convex hull of a finite set of points; a formal polytope is any linear combination of convex polytopes; a convex polytope is degenerate if it is contained in a hyperplane.

Let E be a vector space of dimension n with a given ordered basis $\mathscr{B} = (e_1, \dots, e_n)$. We recall that, if P and Q are two subsets of E, their Minkowski product $P \diamond Q$ is the following set:

$$P \diamond Q := \{t + t' \mid t \in P, t' \in Q\}$$

If P and Q are two convex polytopes, their Minkowski product is a convex polytope.

Definition 1.1. The polytope group $\mathscr{P}(E)$ is the quotient of the free commutative group over the set of all convex polytopes in E by the following relations:

(i) If P and P' are two convex polytopes such that P' is the image of P by a translation with integer coefficients, their classes are equal in $\mathscr{P}(E)$.
(ii) The class in $\mathscr{P}(E)$ of a degenerate convex polytope is 0.
(iii) If $P = \bigcup_{i=1}^{n} P_i$ is a decomposition of a convex polytope P into a union of convex polytopes whose intersections $P_i \cap P_j$, $i \neq j$, are degenerate, the class $[P]$ of P in $\mathscr{P}(E)$ is equal to the sum of the classes of the P_i's.

Figure 1 illustrates the kind of identities which hold in $\mathscr{P}(\mathbb{R}^2)$.

Definition 1.2. The group of n-dimensional simplices of E, $\Delta(E)$, is the subgroup of $\mathscr{P}(E)$ generated by the classes of the simplices:

$$Q_\sigma^E := \{(t_{\sigma(1)}, \dots, t_{\sigma(n)}) \mid 0 \leqslant t_n \leqslant \cdots \leqslant t_1 \leqslant 1\}, \quad \sigma \in S_n.$$

Notice that the definition of $\Delta(E)$ depends on the choice of an ordered basis $\mathscr{B}$ of E. When E is the vector space $\mathbb{R}^n$ with its usual basis, we also use the notation Q_σ^n for Q_σ^E.

An element $[P]$ of $\Delta(E)$ has a unique decomposition

$$[P] = \sum_{\sigma \in S_n} a_\sigma [Q_\sigma^E], \quad a_\sigma \in \mathbb{Z}$$

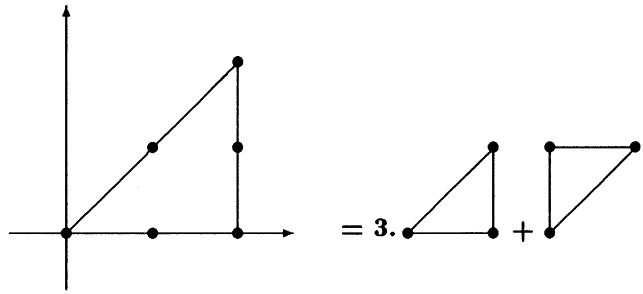

Fig. 1.

besides, $\Delta(E)$ is provided with an involution η defined by

$$\eta([Q^E_\sigma]) := \mathrm{sgn}(\sigma)[Q^E_\sigma]$$

where $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ.

If F is another vector space with an ordered basis $\mathscr{B}'$, the direct sum $E \oplus F$ is provided with the ordered basis defined by the concatenation of $\mathscr{B}$ and $\mathscr{B}'$. The cartesian product $\times$ induces a $\mathbb{Z}$-bilinear map from $\mathscr{P}(E) \otimes \mathscr{P}(F)$ to $\mathscr{P}(E \oplus F)$ which restricts to a $\mathbb{Z}$-bilinear map from $\Delta(E) \otimes \Delta(F)$ to $\Delta(E \oplus F)$ and we define a product $*$ from $\Delta(E) \otimes \Delta(F)$ to $\Delta(E \oplus F)$ by setting

$$* := \eta \circ \times \circ (\eta \otimes \eta).$$

The product $[Q^E_\sigma] * [Q^F_\beta]$ has a simplicial decomposition which is parametrized by the set $Sh(k,l)$ of (k,l)-shuffles (these are the permutations in S_{k+l} such that $\sigma(i) < \sigma(j)$ if $i < j \leqslant k$ or $k < i < j$). Precisely,

$$[Q^E_\sigma] * [Q^F_\beta] = \sum_{\gamma \in Sh(k,l)} \mathrm{sgn}(\gamma)\, [Q^{E \oplus F}_{\gamma \cdot (\sigma|\beta)}]$$

where $(\sigma|\beta)$ is the image of (σ, β) under the embedding $S_k \times S_l \hookrightarrow S_{k+l}$ (see [10] for details on the preceding constructions).

Proposition 1.3 (Patras [10]). *The dilation by k, D^k, $k \in \mathbb{N}^*$, induces a group endomorphism of $\Delta(E)$. We define Ψ^k by $\Psi^k := \eta \circ D^k \circ \eta$, to take into account the orientations. The $\mathbb{Q}$-vector space $\Delta(E) \otimes \mathbb{Q}$ decomposes as a direct sum*

$$\Delta(E) \otimes \mathbb{Q} = \bigoplus_{i=1}^{n} \Delta^{(i)}(E)$$

where Ψ^k acts as k^i on $\Delta^{(i)}(E)$ (the decomposition does not depend on k).

The corresponding projection from $\Delta(E) \otimes \mathbb{Q}$ to $\Delta^{(i)}(E)$ is written $\pi^{(i)}_E$.

Since the dilations commute with cartesian products, we also have the following identity between maps from $\Delta(E) \otimes \Delta(F)$ to $\Delta(E \oplus F)$:

$$\Psi^k \circ * = * \circ (\Psi^k \otimes \Psi^k).$$

Definition 1.4. The geometric boundary $\delta(E)$ of E is the set of all the codimension 1 subspaces of E containing a codimension 1 face of one of the simplices Q_σ^E, $\sigma \in S_n$.

The set $\delta(E)$ does depend on the ordered basis $\mathscr{B}$ and is in bijection with the ordered sequences of vectors of length $n-1$ of the following type:

- $(e_1, \dots, \hat{e}_i, \dots, e_n)$ (where $\hat{e}_i$ means that e_i is omitted).
- $(e_1, \dots, e_{i-1}, e_i + e_j, e_{i+1} \dots, \hat{e}_j, \dots, e_n)$, $i < j$.

Such sequences are ordered basis for the vector spaces $H \in \delta(E)$.

Definition 1.5. The boundary maps $\delta : \Delta(E) \to \bigoplus_{H \in \delta E} \Delta(H)$ are the maps defined on the generators of $\Delta(E)$ by

$$\delta([Q_\sigma^E]) = \left[\left\{\sum_{i \neq \sigma^{-1}(1)} t_{\sigma(i)} e_i \,|\, 0 \leqslant t_n \leqslant \cdots \leqslant t_2 \leqslant 1\right\}\right] + \sum_{i=1}^{n-1} (-1)^i \left[\left\{\sum_{i=1}^{n} t_{\sigma(i)} e_i \,|\, 0 \leqslant t_n \leqslant \cdots \leqslant t_{i+1} = t_i \leqslant \cdots \leqslant t_1 \leqslant 1\right\}\right] + (-1)^n \left[\left\{\sum_{i \neq \sigma^{-1}(n)} t_{\sigma(i)} e_i \,|\, 0 \leqslant t_{n-1} \leqslant \cdots \leqslant t_1 \leqslant 1\right\}\right].$$

Proposition 1.6. *Boundary maps commute with dilations*: $\Psi^k \circ \delta = \delta \circ \Psi^k$. *Moreover, we have:* $\delta \circ \pi_E^{(1)} = \bigoplus_{H \in \delta(E)} \pi_H^{(1)} \circ \delta$.

Indeed, when they act on Euclidean convex polytopes, dilations commute with boundaries. The second identity follows from the first one and from the definition of the $\pi_E^{(i)}$ as projections onto the eigenspaces of Ψ^k (see [10] for details).

We now introduce operations on Euclidean polytopes that we will use in the next section to define operations on cyclic homology theories.

Let $\mathscr{E} = (E, F_1, \dots, F_n)$ be a sequence of vector spaces with ordered basis. We set: $m_i := \dim F_i$, $s(\underline{m}) := m_1 + \cdots + m_n$, $\underline{F} := F_1 \oplus \cdots \oplus F_n$ and $\underline{\Delta}(F) := \Delta(F_1) \otimes \cdots \otimes \Delta(F_n)$.

Definition 1.7. The twisted diagonal map $D_{\mathscr{E}}$ associated to $\mathscr{E}$ is the following map from E to $E \oplus \underline{F}$:

$$D_{\mathscr{E}}(t_1, \dots, t_n) := (t_1, \dots, t_n, \overbrace{t_1, \dots, t_1}^{m_1 \text{ times}}, \overbrace{t_2, \dots, t_2}^{m_2 \text{ times}}, \dots, \overbrace{t_n, \dots, t_n}^{m_n \text{ times}}).$$

For any vector space V, let $\mathscr{P}_V$ be the set of all convex polytopes in V.

Definition 1.8. The twisted cartesian products are the maps $\mathscr{T}_{\mathscr{E}}$ from $\mathscr{P}_E \times \mathscr{P}_{F_1} \times \cdots \times \mathscr{P}_{F_n}$ to $\mathscr{P}_{E \oplus \underline{F}}$ defined by

$$\mathscr{T}_{\mathscr{E}}(P, Q_1, \dots, Q_n) := D_{\mathscr{E}}(P) \diamond (Q_1 \times \cdots \times Q_n)$$

where $Q_1 \times \cdots \times Q_n \in \mathscr{P}_F \hookrightarrow \mathscr{P}_{E \oplus \underline{F}}$.

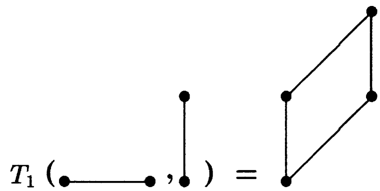

Fig. 2.

Figure 2 illustrates the behaviour of twisted cartesian products in the simplest interesting case: $k = a_1 = 1$.

We define, for any $H_i \in \delta(F_i)$, $E \oplus \underline{F} \ominus H_i \in \delta(E \oplus \underline{F})$ by $E \oplus \underline{F} \ominus H_i := D_{\mathscr{E}}(E) \diamond (F_1 \oplus \cdots \oplus H_i \oplus \cdots \oplus F_n)$.

Proposition – Definition 1.9. *The map $\mathscr{T}_{\mathscr{E}}$ induces a morphism $T_{\mathscr{E}}$ from $\Delta(E) \otimes \underline{\Delta}(F)$ to $\Delta(E \oplus \underline{F})$. Moreover, for any $H \in \delta(E)$ (resp. any $H_i \in \delta(F_i)$), it also induces a morphism, written $T_{\mathscr{E}}^{H \subset E}$ (resp. $T_{\mathscr{E}}^{H_i \subset F_i}$), from $\Delta(H) \otimes \underline{\Delta}(F)$ to $\Delta(H \oplus \underline{F})$ (resp. from $\Delta(E) \otimes \Delta(F_1) \otimes \cdots \otimes \Delta(H_i) \otimes \cdots \otimes \Delta(F_n)$ to $\Delta(E \oplus \underline{F} \ominus H_i)$.*

To take into account orientation phenomena, we set

$$B_{\mathscr{E}} := \eta \circ T_{\mathscr{E}} \circ (\eta \otimes \cdots \otimes \eta)$$

and define maps $B_{\mathscr{E}}^{H \subset E}$ and $B_{\mathscr{E}}^{H_i \subset F_i}$ by the corresponding formulas.

Notice that the maps $T_{\mathscr{E}}^{H \subset E}$ and $T_{\mathscr{E}}^{H_i \subset F_i}$ depend heavily on the embeddings $H \subset E$ and $H_i \subset E_i$.

We will prove that the induced morphism $T_{\mathscr{E}}$ is well defined. The same arguments show that the maps $T_{\mathscr{E}}^{H \subset E}$ and $T_{\mathscr{E}}^{H_i \subset F_i}$ are also well defined. First of all, the map $\mathscr{T}_{\mathscr{E}}$ induces a map $\mathscr{T}'_{\mathscr{E}}$ from $\mathscr{P}(E) \otimes \mathscr{P}(F_1) \otimes \cdots \otimes \mathscr{P}(F_n)$ to $\mathscr{P}(E \oplus \underline{F})$. This follows from the very definitions: the map is compatible with cuttings, glueings and translations. Therefore, we have to prove that, for any $\gamma \in S_n$, $\alpha_i \in S_{m_i}$, the image of $[Q_\gamma^E] \otimes [Q_{\alpha_1}^{F_1}] \otimes \cdots \otimes [Q_{\alpha_n}^{F_n}]$ by $\mathscr{T}'_{\mathscr{E}}$ is contained in the subgroup $\Delta(E \otimes \underline{F})$ of $\mathscr{P}(E \oplus \underline{F})$. That is, we have to show that the class in $P(E \oplus \underline{F})$ of $\mathscr{T}_{\mathscr{E}}(Q_\gamma^E, Q_{\alpha_1}^{F_1}, \ldots, Q_{\alpha_n}^{F_n})$ decomposes into a union of translates of the polytopes $Q_\sigma^{E \oplus \underline{F}}$, $\sigma \in S_{n+s(\underline{m})}$.

In fact $\mathscr{T}'_{\mathscr{E}}([Q_\gamma^E] \otimes [Q_{\alpha_1}^{F_1}] \otimes \cdots \otimes [Q_{\alpha_n}^{F_n}])$ is the class in $\mathscr{P}(E \oplus \underline{F})$ of the polytope P whose underlying set of points is

$$\{(t_{\gamma(1)}, \ldots, t_{\gamma(n)}, t_{\gamma(1)} + u^{(1)}_{\alpha_1(1)}, \ldots, t_{\gamma(1)} + u^{(1)}_{\alpha_1(m_1)}, \ldots, t_{\gamma(n)} + u^{(n)}_{\alpha_n(1)}, \ldots, t_{\gamma(n)} + u^{(n)}_{\alpha_n(m_n)})/$$
$$0 \leqslant t_n \leqslant \cdots \leqslant t_1 \leqslant 1; 0 \leqslant u^{(i)}_{m_i} \leqslant \cdots \leqslant u^{(i)}_1 \leqslant 1, \quad \forall i \in [1,n]\}.$$

These sequences of coordinates satisfy the inequalities:

$$0 \leqslant t_{\gamma(i)} \leqslant t_{\gamma(i)} + u^{(i)}_{m_i} \leqslant \cdots \leqslant t_{\gamma(1)} + u^{(i)}_1 \leqslant 2$$

and we can split P into a union of convex polytopes according to the inequalities

$$0 \leqslant t_{\gamma(i)} \leqslant \cdots \leqslant t_{\gamma(i)} + u^{(i)}_j \leqslant 1 \leqslant t_{\gamma(i)} + u^{(i)}_{j-1} \leqslant \cdots \leqslant t_{\gamma(i)} + u^{(i)}_1 \leqslant 2.$$

This is a disjoint union, up to degenerate polytopes. We have therefore (with the convention that $u^{(i)}_{m_i+1}=0$):

$$P=\bigcup_{j_1\in[1,m_1+1]}\cdots\bigcup_{j_n\in[1,m_n+1]}\{(t_{\gamma(1)},\dots,t_{\gamma(n)},t_{\gamma(1)}+u^{(1)}_{\alpha_1(1)},\dots,t_{\gamma(1)}+u^{(1)}_{\alpha_1(m_1)},\dots,$$
$$t_{\gamma(n)}+u^{(n)}_{\alpha_n(1)},\dots,t_{\gamma(n)}+u^{(n)}_{\alpha_n(m_n)})/0\leqslant t_n\leqslant\cdots\leqslant t_1\leqslant 1;\ 0\leqslant u^{(i)}_{m_i}\leqslant\cdots\leqslant u^{(i)}_1\leqslant 1$$
$$0\leqslant t_{\gamma(i)}\leqslant\cdots\leqslant t_{\gamma(i)}+u^{(i)}_{j_i}\leqslant 1\leqslant t_{\gamma(i)}+u^{(i)}_{j_i-1}\leqslant\cdots\leqslant t_{\gamma(i)}+u^{(i)}_1\leqslant 2\}.$$

Up to a translation with integer coefficients, each of these polytopes identifies with the polytope whose underlying set of points is

$$\{(t_{\gamma(1)},\dots,t_{\gamma(n)},v^{(1)}_{\alpha_1(1)},\dots,v^{(1)}_{\alpha_1(m_1)},\dots,v^{(n)}_{\alpha_n(1)},\dots,v^{(n)}_{\alpha_n(m_n)})/$$
$$0\leqslant t_n\leqslant\cdots\leqslant t_1\leqslant 1;\ 0\leqslant v^{(i)}_{j_i-1}\leqslant\cdots\leqslant v^{(i)}_1\leqslant t_{\gamma(i)}\leqslant\cdots\leqslant v^{(i)}_{j_i}\leqslant 1\}.$$

Such polytopes in turn decompose into a disjoint union of simplices $Q^{E\oplus F}_\sigma$, which is parametrized by all the shuffles of the ordered sequences:

$$0\leqslant v^{(i)}_{j_i-1}\leqslant\cdots\leqslant v^{(i)}_1\leqslant t_{\gamma(i)}\leqslant\cdots\leqslant v^{(i)}_{j_i}\leqslant 1,\quad i=1,\dots,n$$

which respect the ordering $0\leqslant t_n\leqslant\cdots\leqslant t_1\leqslant 1$. This completes the proof. □

These computations yield explicit formulas for $B_{\mathscr{E}}([Q^E_\gamma]\otimes[Q^{F_1}_{\alpha_1}]\otimes\cdots\otimes[Q^{F_n}_{\alpha_n}])$: Writing them down is a tedious but straightforward exercise.

Theorem 1.10 (A_∞-relations for twisted cartesian products)**.** *The following identity holds*:

$$\delta\circ B_{\mathscr{E}}=\left(\sum_{H\in\delta(E)}B^{H\subset E}_{\mathscr{E}}\right)\circ(\delta\otimes 1\otimes\cdots\otimes 1)$$
$$+\sum_{i=1}^{n}\left(\sum_{H_i\in\delta(F_i)}(-1)^{n+\sum_{j=1}^{i-1}m_j}B^{H_i\subset F_i}_{\mathscr{E}}\right)\circ(1\otimes\cdots\otimes\delta\otimes\cdots\otimes 1).$$

Indeed, there is a pointwise correspondence between twisted cartesian products and usual cartesian products given by

$$P\times Q_1\times\cdots\times Q_n\to\mathscr{T}_{\mathscr{E}}(P,Q_1,\dots,Q_n)$$
$$(t_1,\dots,t_n,u^{(1)}_1,\dots,u^{(n)}_{m_n})\mapsto(t_1,\dots,t_n,t_1+u^{(1)}_1,\dots,t_n+u^{(n)}_{m_n}).$$

This correspondence maps the boundary of a cartesian product to the boundary of the corresponding twisted cartesian product, hence the boundary formulas for twisted cartesian products are given by the suitable twisting of boundary formulas for cartesian products.

An explicit proof of the theorem can be given by expressing the polytope $\mathscr{T}_{\mathscr{E}}(Q^E_\gamma,Q^{F_1}_{\alpha_1},\dots,Q^{F_n}_{\alpha_n})$ as a set of points characterized by certain inequalities among the coordinates. Expressing boundary conditions in terms of these inequalities gives the identity.

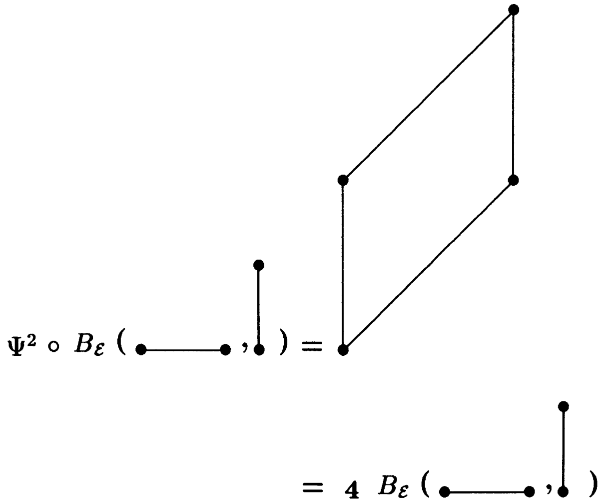

Fig. 3.

Proposition 1.11 (Adams relations for twisted cartesian products). *The following equality holds*:

$$\Psi^k \circ B_{\mathscr{E}} = B_{\mathscr{E}} \circ (\Psi^k \otimes \cdots \otimes \Psi^k).$$

Indeed, the coordinates $(x_1, \dots, x_n, x_1^{(1)}, \dots, x_{m_1}^{(1)}, \dots, x_1^{(n)}, \dots, x_{m_n}^{(n)})$ of a point in $D^k \circ \mathscr{T}_{\mathscr{E}}(P, Q_1, \dots, Q_n)$ must satisfy

$$(x_1, \dots, x_n) \in D^k(P)$$

$$(x_1^{(i)} - x_i, \dots, x_{m_i}^{(i)} - x_i) \in D^k(Q_i)$$

and these relations also characterize the coordinates of the points of the polytope $\mathscr{T}_{\mathscr{E}}(D^k(P), D^k(Q_1), \dots, D^k(Q_n))$. Taking into account orientations gives the expected result.

The proposition is illustrated in Fig. 3 hereafter.

Definition 1.12. The linearization $B_{\mathscr{E},\mathrm{lin}}$ (resp. $B_{\mathscr{E},\mathrm{lin}}^{H \subset E}$; resp. $B_{\mathscr{E},\mathrm{lin}}^{H_i \subset F_i}$), of $B_{\mathscr{E}}$ (resp. $B_{\mathscr{E}}^{H \subset E}$; resp. $B_{\mathscr{E}}^{H_i \subset F_i}$) is the map from $\Delta(E) \otimes \underline{\Delta}(F) \otimes \mathbb{Q}$ to $\Delta(E \oplus \underline{F}) \otimes \mathbb{Q}$ defined by

$$B_{\mathscr{E},\mathrm{lin}} := B_{\mathscr{E}} \circ (\pi_E^{(1)} \otimes 1 \otimes \cdots \otimes 1) \otimes \mathbb{Q}$$

(resp. the map from $\Delta(H) \otimes \underline{\Delta}(F) \otimes \mathbb{Q}$ to $\Delta(H \oplus \underline{F}) \otimes \mathbb{Q}$ defined by

$$B_{\mathscr{E},\mathrm{lin}}^{H \subset E} := (B_{\mathscr{E},\mathrm{lin}}^{H \subset E} \circ (\pi_H^{(1)} \otimes 1 \otimes \cdots \otimes 1)) \otimes \mathbb{Q}$$

resp. the map from $\Delta(E) \otimes \Delta(F_1) \otimes \cdots \otimes \Delta(H_i) \otimes \cdots \otimes \Delta(F_n) \otimes \mathbb{Q}$ to $\Delta(E \oplus \underline{F} \ominus H_i) \otimes \mathbb{Q}$ defined by

$$B_{\mathscr{E},\mathrm{lin}}^{H_i \subset F_i} := (B_{\mathscr{E},\mathrm{lin}}^{H_i \subset F_i} \circ (\pi_E^{(1)} \otimes 1 \otimes \cdots \otimes 1)) \otimes \mathbb{Q}.$$

Proposition 1.13 (Adams and A_∞ relations for the linearization of twisted cartesian products). *We have*

$$\Psi^k \circ B_{\mathscr{E},\mathrm{lin}} = k \cdot B_{\mathscr{E},\mathrm{lin}} \circ (1 \otimes \Psi^k \otimes \cdots \otimes \Psi^k)$$

$$\delta \circ B_{\mathscr{E},\mathrm{lin}} = \left(\sum_{H \in \delta(E)} B_{\mathscr{E},\mathrm{lin}}^{H \subset E} \right) \circ (\delta \otimes 1 \otimes \cdots \otimes 1)$$
$$+ \sum_{i=1}^{n} \left(\sum_{H_i \in \delta(F_i)} (-1)^{n + \sum_{j=1}^{i-1} m_i} B_{\mathscr{E},\mathrm{lin}}^{H_i \subset F_i} \circ (1 \otimes \cdots \otimes \delta \otimes \cdots \otimes 1) \right).$$

The proposition follows from the identities $\delta \circ \pi_E^{(1)} = \bigoplus_{H \in \delta(E)} \pi_H^{(1)} \circ \delta$, $\Psi^k \circ \pi_E^{(1)} = k \cdot \pi_E^{(1)}$ and from the Adams and A_∞ relations (Theorem 1.10 and Proposition 1.11).

We conclude this section by a key lemma, which expresses the values of $T_{\mathscr{E}}^{H \subset E}$. We set $\mathscr{E}_0 := (H, F_2, \ldots, F_n)$ $\mathscr{E}_i := (H, F_1, \ldots, F_i \oplus F_{i+1}, \ldots, F_n)$, $i \in [1, n-1]$ and $\mathscr{E}_n := (H, F_1, \ldots, F_{n-1})$.

Lemma 1.14. *If the ordered basis $\mathscr{H}$ of H is $(\hat{e}_1, e_2, \ldots, e_n)$, then*

$$\mathscr{T}_{\mathscr{E}}(Q_\gamma^H, Q_{\alpha_1}^{F_1}, \ldots, Q_{\alpha_n}^{F_n}) = Q_{\alpha_1}^{F_1} \times \mathscr{T}_{\mathscr{E}_0}(Q_\gamma^H, Q_{\alpha_2}^{F_2}, \ldots, Q_{\alpha_n}^{F_n})$$

if $\mathscr{H} = (e_1, \ldots, e_i + e_{i+1}, \ldots, e_n)$, then

$$\mathscr{T}_{\mathscr{E}}(Q_\gamma^H, Q_{\alpha_1}^{F_1}, \ldots, Q_{\alpha_n}^{F_n}) = \mathscr{T}_{\mathscr{E}_i}(Q_\gamma^H, Q_{\alpha_1}^{F_1}, \ldots, (Q_{\alpha_i}^{F_i} \times Q_{\alpha_{i+1}}^{F_{i+1}}), \ldots, Q_{\alpha_n}^{F_n})$$

and if $\mathscr{H} = (e_1, \ldots, e_{n-1}, \hat{e}_n)$, then

$$\mathscr{T}_{\mathscr{E}}(Q_\gamma^H, Q_{\alpha_1}^{F_1}, \ldots, Q_{\alpha_n}^{F_n}) = \mathscr{T}_{\mathscr{E}_n}(Q_\gamma^H, \ldots, Q_{\alpha_{n-1}}^{F_{n-1}}) \times Q_{\alpha_n}^{F_n}.$$

The lemma can be proved expressing the corresponding polytopes as set of points, as in the proof of Proposition 1.9.

2. Operations on the cyclic bar complexes

Let A be a commutative K-algebra with unit, $\mathbb{Q} \subset K$. Tensor products are taken over K. Set $\bar{A} := A/K$, and let $\bar{C}_*(A) = \oplus_{n \in \mathbb{N}} \bar{C}_n(A) := \oplus_{n \in \mathbb{N}} A \otimes \bar{A}^n$ be the cyclic bar complex of A (see [3]). Set also $C_*^-(A) := \bar{C}_*(A) \otimes K[[u]]$ and $C_*^{\mathrm{per}}(A) := \bar{C}_*(A) \otimes K[u^{-1}, u]]$, where $deg(u) = -2$. These are complexes for the boundary map which comes from the mixed complex structure of the cyclic bar complex [3]. They compute the negative and periodic cyclic homology of A, respectively.

In this section, operations on the cyclic bar complex of A are associated to the formal polytopes constructed in the first part of the paper. Most of the time, the verification that an operation corresponds to a certain polytope relies on a simple cutting and glueing argument. Such verifications are left to the reader.

Let E be again a vector space with an ordered basis $\mathscr{B}$. To each element $[P] \in \Delta(E)$. $[P] = \sum_{\sigma \in S_n} n_\sigma [Q_\sigma^E]$, where $n_\sigma \in \mathbb{Z}$, we associate a linear endomorphism of $\bar{C}_n(A)$, $[\hat{P}]$, defined by

$$[\hat{P}](x_0 \otimes x_1 \otimes \cdots \otimes x_n) := \sum_{\sigma \in S_n} n_\sigma x_0 \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}.$$

Let $H \in \delta(E)$ be provided with the ordered basis $\mathscr{H}$ inherited from the ordered basis $\mathscr{B}$ of E, as in Section 1. Then, any polytope $[P] \in \Delta(H)$ can be written uniquely as $\sum_{\beta \in S_{n-1}} n_\beta [Q_\beta^H]$. We define a linear map $[\hat{P}]_{H \subset E}$ from $C_n(A)$ to $C_{n-1}(A)$ in the following way:

(i) If $\mathscr{H} = (e_1, \ldots, \hat{e}_j, \ldots, e_n)$, the set map:

$$\{1, \ldots, n-1) \to \{1, \ldots, n\}$$
$$k < j \mapsto k$$
$$k \geqslant j \mapsto k+1$$

induces a map $S_{n-1} = Aut_{Ens}\{1, \ldots, n-1\} \to S_n = Aut_{Ens}\{1, \ldots, n\}$, $\sigma \mapsto {}^j\sigma$ whose image leaves j invariant. By definition, $[\hat{P}]_{H \subset E}(x_0 \otimes \cdots \otimes x_n)$ is given by the following sum:

$$\sum_{\beta \in S_{n-1}} n_\beta x_0 x_j \otimes x_{{}^j\beta^{-1}(1)} \otimes \cdots \otimes x_{{}^j\beta^{-1}(j-1)} \otimes x_{{}^j\beta^{-1}(j+1)} \otimes \cdots \otimes x_{{}^j\beta^{-1}(n)}.$$

(ii) If $\mathscr{H} = (e_1, \ldots, e_{i-1}, e_i + e_j, e_{i+1}, \ldots, \hat{e}_j, \ldots, e_n)$, then, with the same notation, $[\hat{P}]_{H \subset E}(x_0 \otimes \cdots \otimes x_n)$ is given by the following sum:

$$\sum_{\beta \in S_{n-1}} n_\beta x_0 \otimes y_{{}^j\beta^{-1}(1)} \otimes \cdots \otimes y_{{}^j\beta^{-1}(j-1)} \otimes y_{{}^j\beta^{-1}(j+1)} \otimes \cdots \otimes y_{{}^j\beta^{-1}(n)}$$

where $y_{{}^j\beta^{-1}(l)} := x_{{}^j\beta^{-1}(l)}$ if ${}^j\beta^{-1}(l) \neq i$ and $y_{{}^j\beta^{-1}(l)} = x_i x_j$ if ${}^j\beta^{-1}(l) = i$.

Example 1. The boundary of $[Q_1^n]$, $\delta[Q_1^n] \in \bigoplus_{H \in \delta(\mathbb{R}^n)} \Delta(H)$ induces a map from $\bar{C}_n(A)$ to $\bar{C}_{n-1}(A)$. It is equal to the usual Hochschild boundary map b (notations are as in [8]). More generally, $b \circ [\widehat{Q_\sigma^n}]$ is equal to the map induced by $\delta[Q_\sigma^n] \in \bigoplus_{H \in \delta(\mathbb{R}^n)} \Delta(H)$.

Example 2. The virtual polytope $(\Psi^k[Q_1^n]) \in \Delta(\mathbb{R}^n)$ induces a linear endomorphism of $\bar{C}_n(A)$. By [10], it is equal to the usual nth Adams operation, also written Ψ^k, on the Hochschild complex. Moreover, for any $\sigma \in S_n$, we have $\Psi^k \circ [\widehat{Q_\sigma^n}] = \widehat{\Psi^k[Q_\sigma^n]}$ [10, Proposition 5].

For any sequence of integers, $\underline{m} = (m_1, \ldots, m_n)$, set $\bar{C}_{\underline{m}}(A) := \bar{C}_{m_1}(A) \otimes \cdots \otimes \bar{C}_{m_n}(A)$. Besides, let $T_{\underline{m}}$ be the map from $\bar{C}_{\underline{m}}(A)$ to $\bar{C}_{s(\underline{m})+n}(A)$ defined by

$$T_{\underline{m}}(1 \otimes x_0^{(1)} \otimes x_1^{(1)} \otimes \cdots \otimes x_{m_n}^{(n)}) := \pm 1 \otimes x_0^{(1)} \otimes \cdots \otimes x_0^{(n)} \otimes x_1^{(1)} \otimes \cdots \otimes x_{m_1}^{(1)} \otimes \cdots \otimes x_{m_n}^{(n)}$$

where the sign on the right-hand side is given by the signature of the corresponding permutation. Finally, let $\mathscr{E} = (E, F_1, \ldots, F_n)$ be as in Section 1.

Definition 2.1. Let $[P] \in \Delta(E \oplus \underline{F})$ (resp. $[P] \in \Delta(H \oplus \underline{F})$, $H \in \delta(E)$; resp. $[P] \in \Delta(E \oplus \underline{F} \ominus H_i)$, $H_i \in \delta(F_i)$).

Let $[P]_{\mathscr{E}}$ be the map from $\bar{C}_{\underline{m}}(A)$ to $\bar{C}_{s(\underline{m})+n}(A)$ defined by $[P]_{\mathscr{E}} := [\hat{P}] \circ T_{\underline{m}}$. We also define maps from $\bar{C}_{\underline{m}}(A)$ to $\bar{C}_{s(\underline{m})+n-1}(A)$ by

$$[P]_{\mathscr{E}}^{H \subset E} := [\hat{P}]_{H \oplus \underline{F} \subset E \oplus \underline{F}} \circ T_{\underline{m}}$$
$$[P]_{\mathscr{E}}^{H_i \subset F_i} := [\hat{P}]_{E \oplus \underline{F} \ominus H_i \subset E \oplus \underline{F}} \circ T_{\underline{m}}.$$

Example 3. If $\mathscr{E} = (\mathbb{R}, \mathbb{R}^n)$ and $[P] = B_{\mathscr{E}}([0,1] \otimes [Q_1^n]) \in \Delta(\mathbb{R} \oplus \mathbb{R}^n)$, the map $[\hat{P}]_{\mathscr{E}}$ from $\bar{C}_n(A)$ to $\bar{C}_{n+1}(A)$ is equal to the normalized Connes' boundary map $\bar{B}$ (see e.g. [8], 2.1.9 for a definition of $\bar{B}$).

Example 4. If $\mathscr{E} = (\mathbb{R}^2, \mathbb{R}^n, \mathbb{R}^m)$ and $[P] = B_{\mathscr{E}}([Q_1^2] \otimes [Q_1^n] \otimes [Q_1^m]) \in \Delta(\mathbb{R}^2 \oplus \mathbb{R}^n \oplus \mathbb{R}^m)$, then $[P]_{\mathscr{E}}$ is equal to the Getzler–Jones and Loday cyclic shuffle map [3, pp. 279; 8, pp. 127].

More generally, if $\mathscr{E} = (\mathbb{R}^k, \mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_k})$, and $[P] = B_{\mathscr{E}}([Q_1^k] \otimes \cdots \otimes [Q_1^{m_k}])$, then $[P]_{\mathscr{E}}$ is equal to the corresponding Getzler–Jones (higher) cyclic shuffle operator on the Hochschild complexes [3, pp. 279].

Lemma 2.2. *Let $[P]$ belong to $\Delta(E \oplus \underline{F})$. We also assume that $\delta[P] \in (\bigoplus_{H \in \delta(E)} \Delta(H \oplus \underline{F})) \oplus (\bigoplus_{i \in [1,n]}(\bigoplus_{H_i \in \delta(F_i)} \Delta(E \oplus \underline{F} \ominus H_i)))$ and write $\delta[P]$ for the induced map from $\bar{C}_{\underline{m}}(A)$ to $\bar{C}_{s(\underline{m})+n-1}(A)$. Then $b \circ [P]_{\mathscr{E}} = \delta[P]$.*

The identity follows from the equalities in Example 1.

Lemma 2.3. *Let $[P]$ belong to $\Delta(E \oplus \underline{F})$, then $\Psi^k \circ [P]_{\mathscr{E}} = (\Psi^k[P])_{\mathscr{E}}$.*

Example 5. It follows from this lemma and from Example 3 that Connes' boundary map (almost) commutes with Adams operations. Namely,

$$\Psi^k \circ \bar{B} = k\bar{B} \circ \Psi^k.$$

Lemma 2.4. *If $\mathscr{E} = (\mathbb{R}^n, \mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_n})$, then, the following identity holds between morphisms from $\bar{C}_{\underline{m}}(A)$ to $\bar{C}_{s(\underline{m})+n}$:*

$$(B_{\mathscr{E}}([Q_1^n] \otimes \Psi^{l_1}([Q_1^{m_1}]) \otimes \Psi^{l_2}([Q_1^{m_2}]) \otimes \cdots \otimes \Psi^{l_n}([Q_1^{m_n}]))_{\mathscr{E}}$$
$$= (B_{\mathscr{E}}([Q_1^n] \otimes [Q_1^{m_1}] \otimes \cdots \otimes [Q_1^{m_n}]))_{\mathscr{E}} \circ (\Psi^{l_1} \otimes \cdots \otimes \Psi^{l_n}).$$

Lemma 2.3 is a consequence of [10], Proposition 5 (see also Example 2). Lemma 2.4 is proven by the same kind of computations.

Definition 2.5. Let $\mu_n : (\bar{C}_*(A))^{\otimes n} \to \bar{C}_*(A)$ be the multilinear map of degree n defined by

$$\forall \underline{m} = (m_1, \dots, m_n),\ \mu_{n|\bar{C}_{\underline{m}}(A)} := (B_{\mathscr{E},\mathrm{lin}}([Q_1^n] \otimes \cdots \otimes [Q_1^{m_n}]))_{\mathscr{E}}$$

where $\mathscr{E} := (\mathbb{R}^n, \mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_n})$.

Proposition 2.6 (Adams relations for the μ_n's)**.** *We have*

$$\Psi^k \circ \mu_n = k \cdot \mu_n \circ (\Psi^k \otimes \cdots \otimes \Psi^k).$$

This identity follows from the Adams relations for the linearizations of twisted cartesian products.

We write ω for the shuffle product on the reduced Hochschild complex [8, 4.2].

Proposition 2.7 (A_∞ relations for the μ_n's). *On $\bar{C}_{\underline{m}}(A)$ we have*

$$b\circ\mu_n=(-1)^{(n-1)m_1}\omega\circ(Id\otimes\mu_{n-1})+\sum_{i=1}^{n-1}(-1)^i\mu_{n-1}(Id\otimes\cdots\otimes\omega\otimes\cdots\otimes Id)$$

$$+(-1)^n\omega\circ(\mu_{n-1}\otimes Id)+\sum_{i=1}^{n}(-1)^{n+m_1+\cdots+m_{i-1}}\mu_n(1\otimes\cdots\otimes b\otimes\cdots\otimes 1).$$

The identity follows from Proposition 1.13, Lemma 1.14, from the combinatorial description of the product $*$ in Section 1 by means of shuffles and from Lemma 2.2.

Definition 2.8 (A_∞ operations on the negative and periodic cyclic complexes)**.** Let m_n, $n\in\mathbb{N}$ be the $K[[u]]$-multilinear (resp. $K[u^{-1},u]]$-multilinear) operation from $(\bar{C}_*(A)\otimes K[[u]])^{\otimes n}$ to $\bar{C}_*(A)\otimes K[[u]]$ (resp. replacing $K[[u]]$ by $K[u^{-1},u]]$) defined by

$$m_1:=b+u\mu_1$$

$$m_2:=\omega+u\mu_2$$

$$m_n:=u\mu_n.$$

Observations. m_1 is the usual boundary map on the negative and periodic cyclic complexes, since $\mu_1=\bar{B}$ (Example 3). The product m_2 is different from the Getzler–Jones and Loday product on negative and periodic cyclic complexes [3, 8], as can be easily checked (this property also follows from the description of μ_2 in Proposition 2.12).

Let us recall from [12] or [3] the definition of A_∞-algebras. These are differential graded algebras which are associative up to a coherent family of higher homotopies. Precisely, a graded vector space V is an A_∞-algebra if and only if it is provided with a family of multilinear maps $m_n: V^{\otimes n}\to V$ of degrees $n-2$ satisfying for each n the relations

$$\sum_{i+j=n+1}\sum_{l=1}^{i}(-1)^{\varepsilon(l,j)}m_i(a_1,\ldots,a_{l-1},m_j(a_l,\ldots,a_{l+j-1}),a_{l+j},\ldots,a_n)=0$$

where $\varepsilon(l,j):=(j+1)l+j(n+\sum_{p=1}^{l-1}deg(a_p))$.

Theorem 2.9. *The maps m_n define an A_∞-algebra structure on $\bar{C}_*(A)\otimes K[[u]]$ (resp. $\bar{C}_*(A)\otimes K[u^{-1},u]]$). The Adams operations Ψ^k are A_∞-algebra endomorphisms.*

We recall that the action of the Adams operation Ψ^n on $\bar{C}_*(A)$ is extended to the cyclic bar complex $\bar{C}_*(A)\otimes K[[u]]$ by setting: $\Psi^n(u):=n^{-1}u$ [8]. Since we are dealing with *reduced* Hochschild and cyclic complexes, by the very definition of the maps μ_n, we have

$$\mu_n\circ(Id\otimes\cdots\otimes\mu_l\otimes\cdots\otimes Id)=0,\quad\forall n,l.$$

The relations of an A_∞-algebra are then satisfied by the m_n, according to Proposition 2.7. The Adams operations are A_∞-algebra endomorphisms by Proposition 2.6.

This theorem implies, as a corollary, the compatibility with the products of the weight decompositions of the various cyclic homologies of a commutative algebra over the rationals (compare with [1, 7–9]).

We conclude by giving a combinatorial description of the μ_n's.

We recall from [10] that the projection $\Delta(\mathbb{R}^n)\otimes\mathbb{Q}\to\Delta^{(1)}(\mathbb{R}^n)$ is given by the first Eulerian idempotent $e_n^1\in\mathbb{Q}[S_n]$ (see [11] for details on this idempotent). The rational coefficients of e_n^1 in its expansion in the canonical basis of $\mathbb{Q}[S_n]$ are written c_σ

$$e_n^1=\sum_{\sigma\in S_n}c_\sigma\sigma.$$

Definition 2.10 (Cyclic shuffles with symmetries). For any n-uple $\underline{m}=(m_1,\dots,m_n)$ and any $\sigma\in S_n$, we define the cyclic shuffle with symmetry σ, $\tau_{(\sigma,\underline{m})}\in\mathbb{Z}[S_{n+s(\underline{m})}]$ as follows. By definition, $\mathrm{sgn}(\sigma)\cdot\tau_{(\sigma,\underline{m})}$ is the sum over all the signed permutations $\beta\in S_{n+s(\underline{m})}=Aut_{Ens}\{1,\dots,n+s(\underline{m})\}$ such that

(i) The ordered sequence $\beta(\sigma^{-1}(1)),\dots,\beta(\sigma^{-1}(n))$ is increasing;

(ii) For $i=1,\dots,n$, the ordered sequences $(\beta(i),\beta(n+m_1+\cdots+m_i),\dots,\beta(n+m_1+\cdots+m_{i-1}+1))$ are increasing up to any cyclic reordering (that is, either this sequence is increasing, either for some $j\in[1,m_i]$ the sequence $\beta(n+m_1+\cdots+m_{i-1}+j),\dots,\beta(n+m_1+\cdots+m_{i-1}+1),\beta(i),\beta(n+m_1+\cdots+m_i),\dots,\beta(n+m_1+\cdots+m_{i-1}+j+1)$ is increasing).

Let us define an action of $\mathbb{Q}[S_i]$ on $A\otimes\bar{A}^{\otimes i}$ by

$$\sigma(x_0\otimes\cdots\otimes x_i):=\mathrm{sgn}(\sigma)x_0\otimes x_{\sigma^{-1}(1)}\otimes\cdots\otimes x_{\sigma^{-1}(n)}.$$

Proposition 2.11. *The action of μ_n on $\bar{C}_{\underline{m}}(A)$ is given by the following identity*:

$$\mu_n=\sum_{\sigma\in S_n}c_\sigma\tau_{(\sigma,\underline{m})}\circ T_{\underline{m}}.$$

Indeed, by definition, if $\mathscr{E}=(\mathbb{R}^n,\mathbb{R}^{m_1},\dots,\mathbb{R}^{m_n})$, then

$$\mu_{n|\bar{C}_{\underline{m}}(A)}=\left(B_{\mathscr{E}}\left(\left(\left(\sum_{\sigma\in S_n}c_\sigma\cdot[Q_\sigma^n]\right)\otimes[Q_1^{m_1}]\right)\otimes\cdots\otimes[Q_1^{m_n}]\right)\right)_{\mathscr{E}}$$

$$=\left(\sum_{\sigma\in S_n}c_n\cdot B_{\mathscr{E}}([Q_\sigma^n]\otimes[Q_1^{m_1}]\otimes\cdots\otimes[Q_1^{m_n}])\right)_{\mathscr{E}}.$$

The geometric decomposition of these polytopes into a union of simplices gives the expected result.

Proposition 2.12. *The A_∞-algebra structure of Theorem 2.9 induces the Hood–Jones product in negative and periodic cyclic homology* (*resp. the Loday–Quillen product in cyclic homology*).

Since $e_2^1=(1-\sigma)/2$, where σ is the non-trivial permutation in S_2, we get from Proposition 2.11 that $\mu_2=\pm B_2\circ(1+\sigma)/2$. Here B_2 is Getzler–Jones' cyclic shuffle map [3, p. 279] and the action of

$\sigma \in S_2$ on $\bar{C}_n(A) \otimes \bar{C}_m(A)$ is given by $\sigma(x \otimes y) := (-1)^{n \cdot m} y \otimes x$. The proposition follows from the graded commutativity of the products in negative, periodic and usual cyclic homology [6].

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